Finding the tetrahedron circumcenter

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The circumcenter is the center of the circumsphere – the sphere on which all 4 tetrahedron corners lie.

Wikipedia states that you can compute the circumcenter C of a tetrahedron with corners x_0, x_1, x_2, x_3 using:

$$C = A^{-1}B \quad \text{where} \quad A = \begin{pmatrix} [x_1 - x_0]^{\mathrm{T}} \\ [x_2 - x_0]^{\mathrm{T}} \\ [x_3 - x_0]^{\mathrm{T}} \end{pmatrix} \quad \text{and} \quad B = \frac{1}{2}\begin{pmatrix} x_1^2 - x_0^2 \\ x_2^2 - x_0^2 \\ x_3^2 - x_0^2 \end{pmatrix}$$

Let us derive this.

First, consider the fact that the equation of a plane with normal $\vec{n} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$ and one known point $\vec{p} = \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$ is the following (all points $\vec{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are on the plane):

$$\vec{n} \cdot (\vec{X} - \vec{p}) = 0$$

which means that any vector in the plane is orthogonal to the plane's normal vector.

Multiplying out we obtain the second common way of writing that equation (mentioned here only for familiarity; we won't use this form further):

$$\begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \cdot \left(\begin{pmatrix} x \\ y \\ z \end{pmatrix} - \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix} \right) = 0 \quad \Leftrightarrow \quad n_x(x - p_x) + n_y(y - p_y) + n_y(x - p_y) = 0$$

Rearranging the vector form so that the unknown-vector $\vec{X}$ stands as alone as possible (the dot product is distributive):

$$\vec{n} \cdot \vec{X} = \vec{n} \cdot \vec{p}$$

If we want to intersect 3 planes of that form, we get 3 equations constraining the same $\vec{X}$ which will be the intersection point:

$$\begin{aligned} \vec{n}_1 \cdot \vec{X} &= \vec{n}_1 \cdot \vec{p}_1 \\ \vec{n}_2 \cdot \vec{X} &= \vec{n}_2 \cdot \vec{p}_1 \\ \vec{n}_3 \cdot \vec{X} &= \vec{n}_3 \cdot \vec{p}_1 \end{aligned}$$

We can write these as one matrix equation:

$$\begin{pmatrix} \boxed{\vec{n}_1{}^T} \\ \boxed{\vec{n}_2{}^T} \\ \boxed{\vec{n}_3{}^T} \end{pmatrix} \vec{X} = \begin{pmatrix} \vec{n}_1 \cdot \vec{p}_1 \\ \vec{n}_2 \cdot \vec{p}_2 \\ \vec{n}_3 \cdot \vec{p}_3 \end{pmatrix}$$

We can solve for $\vec{X}$ my left-multiplying with the inverse of the leftmost matrix:

$$\vec{X} = \begin{pmatrix} \boxed{\vec{n_1}^T} \\ \boxed{\vec{n_2}^T} \\ \boxed{\vec{n_3}^T} \end{pmatrix}^{-1} \begin{pmatrix} \vec{n_1} \cdot \vec{p_1} \\ \vec{n_2} \cdot \vec{p_2} \\ \vec{n_3} \cdot \vec{p_3} \end{pmatrix}$$

Now to the tetrahedron.

The circumcenter is the intersection between three bisector planes. (A bisector plane of a line is the plane orthogonal to the line, cutting through is center.) The three bisected lines must not be all on the same face of the tetrahedtron, but instead must "span" the tetrahedron.

Let's call the 4 tetrahedron vertices corner points A, B, C, D.

The angle bisector of line $\overrightarrow{\mathrm{AB}} = [B - A]$ is a plane with normal $\vec{n} = \overrightarrow{\mathrm{AB}}$ itself, and the line's middle point $\vec{A} + \frac{\overrightarrow{\mathrm{AB}}}{2} = A + \frac{[B - A]}{2} = \frac{2A + B - A}{2} = \frac{A + B}{2}$ on it:

$$\overrightarrow{\mathrm{AB}} \cdot \left(\vec{X} - \frac{A + B}{2} \right) = 0$$

The equivalent applies for lines $\overrightarrow{\mathrm{AC}}$ and $\overrightarrow{\mathrm{AD}}$.

Intersecting the three planes, plugging their normal vectors and points into our equation for the plane intersection solution from above, we get:

$$\vec{X} = \begin{pmatrix} \boxed{\overrightarrow{\mathrm{AB}}^T} \\ \boxed{\overrightarrow{\mathrm{AC}}^T} \\ \boxed{\overrightarrow{\mathrm{AD}}^T} \end{pmatrix}^{-1} \begin{pmatrix} \overrightarrow{\mathrm{AB}} \cdot \frac{A+B}{2} \\ \overrightarrow{\mathrm{AC}} \cdot \frac{A+C}{2} \\ \overrightarrow{\mathrm{AD}} \cdot \frac{A+C}{2} \end{pmatrix}$$

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$$\Leftrightarrow \vec{X} = \begin{pmatrix} \boxed{[B - A]^T} \\ \boxed{[C - A]^T} \\ \boxed{[D - A]^T} \end{pmatrix}^{-1} \frac{1}{2} \begin{pmatrix} [B + A] \cdot [B - A] \\ [C + A] \cdot [C - A] \\ [D + A] \cdot [D - A] \end{pmatrix}$$

now $(a + b)(a - b) = a^2 - b^2$

$$\Leftrightarrow \quad \vec{X} = \begin{pmatrix} \boxed{[B - A]^T} \\ \boxed{[C - A]^T} \\ \boxed{[D - A]^T} \end{pmatrix}^{-1} \frac{1}{2} \begin{pmatrix} B^2 - A^2 \\ C^2 - A^2 \\ D^2 - A^2 \end{pmatrix}$$

which is exactly the original formula from Wikipedia, with A, B, C, D renamed to x_0, x_1, x_2, x_3 and $\vec{X}$ being the circumcenter C.